AP Calculus AB
Lesson 6-3 *Definite Integrals & Antiderivatives*, Part 1

Name Heinl 2017
Date ______

Learning Goal:

- *I can apply rules for definite integrals.*

In the previous section, we defined $\int_a^b f(x)$ as a limit of sums $\sum c_k \Delta x_k$. In doing so, we moved from *left to right* across the interval $[a,b]$, so $x_k - x_{k-1}$ is always greater than zero. What would happen if we integrated in the *opposite direction*?

If we integrated in the opposite direction, the integral $\int_a^b f(x)$ would become $\int_b^a f(x)$. The integral $\int_b^a f(x)$ is again a sum in the form $\sum c_k \Delta x_k$, but this time Δx_k would become $x_{k-1} - x_k$, which would be [redacted] than zero. This would change the sign on all the terms of the Riemann sum, and ultimately the sign of the definite integral. This idea suggests

$$\boxed{\int_b^a f(x) = -\int_a^b f(x)}$$

Along with the above rule, there are several other rules for dealing with definite integrals. They are listed below (*and will need to be memorized!*)

Rules for Definite Integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. *Zero:* $\int_a^a f(x)\,dx = 0$

3. *Constant Multiple:* $\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a,b]$, then
$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$ [example on next page]

7. *Domination:* $f(x) \ge 0$ on $[a,b] \Rightarrow \int_a^b f(x) \ge 0$
$f(x) \ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x) \ge \int_a^b g(x)$

8. *Unsigned Area:* Since $\int_a^b f(x)\,dx$ gives the "signed" area under the curve, then
$\int_a^b |f(x)|\,dx$ gives the total "unsigned" area of $f(x)$

OVER →

Practice

Suppose: $\int_{-1}^{1} f(x)\,dx = 5$, $\int_{1}^{4} f(x)\,dx = -2$, and $\int_{-1}^{1} h(x)\,dx = 7$

Find each of the following integrals, if possible. Show your set-up using the above information.

(a) $\int_{4}^{1} f(x)\,dx$

order!

$= -\int_{1}^{4} f(x)\,dx$

$= -1 \cdot -2$

$= \boxed{2}$

(b) $\int_{-1}^{4} f(x)\,dx$ Additivity

$\int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = \int_{-1}^{4} f(x)\,dx$

$5 + -2 = \boxed{3}$

(c) $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx$

$2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx =$

$2 \cdot 5 + 3 \cdot 7 =$

$10 + 21 =$

$\boxed{31}$

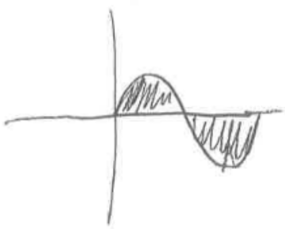
(d) $\int_{-2}^{2} h(x)\,dx$

Not possible w/given info.

(e) Without your calculator, show that the value of $\int_{0}^{2} \sqrt{1+\cos x}\,dx$ is less than 3 (*hint: use rule # 6 above*) → Max-min

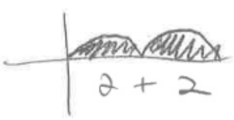
The maximum value of $1+\cos x$ is 2. So the maximum value of $\sqrt{1+\cos x}$ is $\sqrt{2}$. By the max-min inequality, the maximum of $\int_0^2 \sqrt{1+\cos x}$ is $\sqrt{2}(2-0) = 2\sqrt{2} \approx 2.828$ $\therefore \int_0^2 \sqrt{1+\cos x} < 3$

(f) Recall that $\int_{0}^{\pi} \sin x = 2$. Without the assistance of your calculator, find $\int_{0}^{2\pi} |\sin x|$. Verify your answer on your calculator.



$\int_0^{2\pi} \sin x = 0$. But since we want the "unsigned" area, $\int_0^{2\pi} |\sin x| = 4$

Think:



AP Calculus AB
Lesson 6-3: *Definite Integrals & Antiderivatives*, Part 2

Name Heinl 2017
Date

Learning Goal:

- *I can find the average value of a function over a closed interval.*

Review

Let $\int_{-1}^{5} f(x)\,dx = -3$, $\int_{5}^{7} f(x)\,dx = 4$, $\int_{5}^{7} h(x)\,dx = 7$. Evaluate the below integrals, if possible.

1. $\int_{7}^{5} f(x)\,dx$

$-\int_{5}^{7} f(x)\,dx =$

$-1 \cdot 4 = \boxed{-4}$

2. $\int_{-1}^{7} f(x)\,dx$

$\int_{-1}^{5} f(x)\,dx + \int_{5}^{7} f(x)\,dx =$

$-3 + 4 = \boxed{1}$

3. $\int_{10}^{14} h(x)\,dx$

Not enough info to evaluate.

4. $\int_{5}^{7} [2 \cdot f(x) - 3 \cdot h(x)]\,dx$

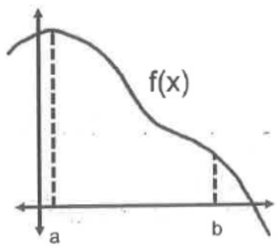
$2\int_{5}^{7} f(x)\,dx - 3\int_{5}^{7} h(x)\,dx =$

$2 \cdot 4 - 3 \cdot 7 =$

$8 - 21 = \boxed{-13}$

Average Value of a Function

The average of n numbers is simply the sum of the numbers divided by n. How would we define the average value of an arbitrary function f over a closed interval $[a,b]$? The problem is that there is an infinite number of y values for the function, and we cannot divide by infinity. The solution is to look at just a "large" number of values in $[a,b]$.



Think about dividing $[a,b]$ into n equal subintervals, where n is an arbitrary "large" number. Then the length of each subinterval would be $\Delta x = \frac{b-a}{n}$. Remember, to find the area under the curve from $[a,b]$, we chose some value c_k in each interval Δx.

The average of the n sampled values is $\dfrac{f(c_1)+f(c_2)+\ldots+f(c_n)}{n} = \dfrac{1}{n}\sum_{k=1}^{n} f(c_k)$

Remember that $\Delta x = \dfrac{b-a}{n}$. Rearranging this equation, we get $\dfrac{1}{n} = \dfrac{\Delta x}{b-a}$

Therefore $\dfrac{1}{n}\sum_{k=1}^{n} f(c_k) = \dfrac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k)$

$$\boxed{= \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\,\Delta x}$$ Average Value

OVER →

Does the last line look familiar? It should. It is $\frac{1}{b-a}$ times a Riemann sum for f on $[a,b]$.

This means that when we consider this averaging process as $n \to \infty$, we find it has a limit and the limit is

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

Wow!! This exciting result means that the *average value of a function* can be found with a definite integral (note that this is a popular topic on the AP Exam – but do not confuse it with average rate of change!)

Average (Mean) Value

If f is integrable on $[a,b]$, its **average (mean) value** on $[a,b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Practice

Use your calculator to find the average value of $f(x) = 4 - x^2$ on $[0,3]$

$$av(f) = \frac{1}{3-0}\int_0^3 (4-x^2)\,dx$$

$$= \frac{1}{3}\int_0^3 (4-x^2)\,dx = \boxed{1}$$

The Mean Value Theorem for Definite Integrals

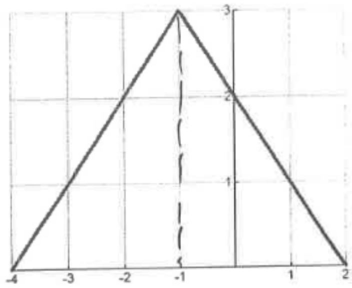
If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Together with the Mean Value Theorem for derivatives, this means that if a function f is continuous on $[a,b]$ and differentiable on (a,b), then there is some point in $[a,b]$ where the derivative equals the average rate of change over $[a,b]$ and another point on $[a,b]$ where the function equals its average value.

Practice

1. Find the average value of
$$f(x)=\begin{cases} x+4, & -4\le x\le -1 \\ -x+2, & -1\le x\le 2 \end{cases} \text{ on } [-4,2].$$
Use the graph at the right to help.



$$av(f) = \frac{1}{2-(-4)}\int_{-4}^{2} f(x)\,dx =$$

$$\frac{1}{6}\int_{-4}^{2} f(x)\,dx =$$

$$\frac{1}{6}\cdot\left[\tfrac{1}{2}\cdot 6\cdot 3\right] =$$

$$\boxed{\frac{3}{2}}$$

2. If $\int_1^5 f(x)dx = 15$ find the average value of $f(x)$ on the interval $[1, 5]$.

A 3 $\qquad$ $\boxed{C \ \frac{15}{4}}$

B 5 $\qquad$ D $\frac{15}{2}$

$$av(f) = \frac{1}{5-1}\cdot 15 = \frac{15}{4}$$